Grassmannian Frames with Applications to Coding and Communication

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Abstract

For a given class $\mathcal{F}$ of unit norm frames of fixed redundancy we define a Grassmannian frame as one that minimizes the maximal correlation $|\langle f_k, f_l \rangle|$ among all frames $\{f_k\}_{k\in\mathcal{I}} \in \mathcal{F}$. We first analyze finite-dimensional Grassmannian frames. Using links to packings in Grassmannian spaces and antipodal spherical codes we derive bounds on the minimal achievable correlation for Grassmannian frames. These bounds yield a simple condition under which Grassmannian frames coincide with unit norm tight frames. We exploit connections to graph theory, equiangular line sets, and coding theory in order to derive explicit constructions of Grassmannian frames. Our findings extend recent results on unit norm tight frames. We then introduce infinite-dimensional Grassmannian frames and analyze their connection to unit norm tight frames for frames which are generated by group-like unitary systems. We derive an example of a Grassmannian Gabor frame by using connections to sphere packing theory. Finally we discuss the application of Grassmannian frames to wireless communication and to multiple description coding.

Key words: Frame, Grassmannian spaces, spherical codes, Gabor frame, multiple description coding, unit norm tight frame, conference matrix, equiangular line sets, unitary system.

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1 Introduction

Orthonormal bases are an ubiquitous and eminently powerful tool that pervades all areas of mathematics. Sometimes however we find ourselves in a situation where a representation of a function or an operator by an overcomplete spanning system is preferable over the use of an orthonormal basis. One reason for this may be that an orthonormal basis with the desired properties does not exist. A classical example occurs in Gabor analysis, where the Balian-Low theorem tells us that orthonormal Gabor bases with good time-frequency localization cannot exist, while it is not difficult to find overcomplete Gabor systems with excellent time-frequency localization. Another important reason is the deliberate introduction of redundancy for the purpose of error correction in coding theory.

When dealing with overcomplete spanning systems one is naturally lead to the concept of *frames* [11]. Recall that a sequence of functions $\{f_k\}_{k\in\mathcal{I}}$ ($\mathcal{I}$ is a countable index set) belonging to a separable Hilbert space $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist positive constants (*frame bounds)* A and B such that

$$A\|f\|_2^2 \le \sum_{k\in\mathcal{I}} |\langle f, f_k\rangle|^2 \le B\|f\|_2^2 \tag{1}$$

for every $f \in \mathcal{H}$.

Even when there are good reasons to trade orthonormal bases for frames we still want to preserve as many properties of orthonormal bases as possible. There are many equivalent conditions to define an orthonormal basis $\{e_k\}_{k\in\mathcal{I}}$ for $\mathcal{H}$, such as

$$f = \sum_{k\in\mathcal{I}} \langle f, e_k\rangle e_k, \quad \forall f \in \mathcal{H}, \quad \text{and } \|e_k\| = 1, \ \forall k \in \mathcal{I}, \tag{2}$$

or

$$\{e_k\}_{k\in\mathcal{I}} \text{ is complete in } \mathcal{H} \text{ and } \langle e_k, e_l\rangle = \delta_{k,l}, \tag{3}$$

where $\delta_{k,l}$ denotes the Kronecker delta.

These two definitions suggest two ways to construct frames that are "as close as possible" to orthonormal bases. Focusing on condition (2) we are naturally lead to unit norm tight frames, which satisfy

$$f = \frac{1}{A}\sum_{k\in\mathcal{I}} \langle f, f_k\rangle f_k, \ \forall f \in \mathcal{H}, \quad \text{and } \|f_k\|_2 = 1, \ \forall k \in \mathcal{I}, \tag{4}$$

where $A = B$ are the frame bounds. This class of frames has been frequently studied and is fairly well understood [11, 7, 24, 29, 21].

As an alternative, as proposed in this paper, we focus on condition (3), which essentially states that the elements of an orthonormal basis are perfectly uncorrelated. This suggests to search for frames $\{f_k\}_{k \in \mathcal{I}}$ such that the maximal correlation $|\langle f_k, f_l \rangle|$ for all $k, l \in \mathcal{I}$ with $k \neq l$, is as small as possible. This idea will lead us to so-called *Grassmannian frames*, which are characterized by the property that the frame elements have minimal cross-correlation among a given class of frames. The name "Grassmannian frames" is motivated by the fact that in finite dimensions Grassmannian frames coincide with optimal packings in certain Grassmannian spaces as we will see in Section 2.

Recent literature on finite-dimensional frames [21, 7, 14] indicates that the connection between finite frames and areas such as spherical codes, algebraic geometry, graph theory, and sphere packings is not well known in the "frame community". This has led to a number of rediscoveries of classical constructions and duplicate results. The concept of Grassmannian frames will allow us to make many of these connections transparent.

The paper is organized as follows. In the remainder of this section we introduce some notation used throughout the paper. In Section 2 we focus on finite Grassmannian frames. By utilizing a link to spherical codes and algebraic geometry we derive lower bounds on the minimal achievable correlation between frame elements depending on the redundancy of the frame. We further show that optimal finite Grassmannian frames which achieve this bound are also tight and certain unit norm tight frames are also Grassmannian frames. We discuss related concepts arising in graph theory, algebraic geometry and coding theory and provide explicit constructions of finite Grassmannian frames. In Section 3 we extend the concept of Grassmannian frames to infinite-dimensional Hilbert spaces and analyze the connection to unit norm tight frames. We give an example of a Grassmannian frame arising in Gabor analysis. Finally, in Section 4 we discuss applications in multiple description coding theory.

1.1 Notation

We introduce some notation and definitions used throughout the paper. Let $\{f_k\}_{k \in \mathcal{I}}$ be a frame for a finite- or infinite-dimensional Hilbert space $\mathcal{H}$. Here

$\mathcal{I}$ is an index set such as $\mathbb{Z}, \mathbb{N}$ or $\{0, \ldots, N-1\}$. The *frame operator* S associated with the frame $\{f_k\}_{k\in\mathcal{I}}$ is defined by

$$Sf = \sum_{k\in\mathcal{I}} \langle f, f_k \rangle f_k. \tag{5}$$

S is a positive definite, invertible operator that satisfies $AI \le S \le BI$, where I is the identity operator on $\mathcal{H}$. The *frame analysis operator* $T : \mathcal{H} \to \ell_2(\mathcal{I})$ is given by

$$Tf = \{\langle f, f_k \rangle\}_{k\in\mathcal{I}}, \tag{6}$$

and the *frame synthesis operator* is

$$T^* : \ell_2(\mathcal{I}) \to \mathcal{H} : T\{c_k\}_{k\in\mathcal{I}} = \sum_{k\in\mathcal{I}} c_k f_k. \tag{7}$$

Any $f \in \mathcal{H}$ can be expressed as

$$f = \sum_{k\in\mathcal{I}} \langle f, f_k \rangle h_k = \sum_{k\in\mathcal{I}} \langle f, h_k \rangle f_k, \tag{8}$$

where $\{h_k\}_{k\in\mathcal{I}}$ is the *canonical dual frame* given by $h_k = S^{-1} f_k$. If $A = B$ the frame is called *tight*, in which case $S = AI$ and $h_k = \frac{1}{A} f_k$. The tight frame canonically associated to $\{f_k\}_{k\in I}$ is $S^{-\frac{1}{2}} f_k$.

If $\|f_k\| = 1$ for all k then $\{f_k\}_{k\in\mathcal{I}}$ is called a *unit norm frame*. Here $\|.\|$ denotes the ℓ_2-norm of a vector in the corresponding finite- or infinite-dimensional Hilbert space. *Unit norm tight frames* have many nice properties which make them an important tool in theory [36, 26] and in a variety of applications [21, 43, 42, 15]. Observe that if $\{f_k\}_{k\in\mathcal{I}}$ is a unit norm frame, then $\{S^{-\frac{1}{2}} f_k\}_{k\in\mathcal{I}}$ is a tight frame, but in general no longer of unit norm type!

We call a unit norm frame $\{f_k\}_{k\in\mathcal{I}}$ *equiangular* if

$$|\langle f_k, f_l \rangle| = c \quad \text{for all } k, l \text{ with } k \ne l, \tag{9}$$

for some constant $c \ge 0$. Obviously any orthonormal basis is equiangular.

2 Finite Grassmannian frames, spherical codes, and equiangular lines

In this section we concentrate on frames $\{f_k\}_{k=1}^N$ for $\mathbb{E}^m$ where $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$. As mentioned in the introduction we want to construct frames $\{f_k\}_{k=1}^N$ such that the maximal correlation $|\langle f_k, f_l\rangle|$ for all $k, l \in \mathcal{I}$ with $k \neq l$, is as small as possible. If we do not impose any other conditions on the frame we can set $N = m$ and take $\{f_k\}_{k=1}^N$ to be an orthonormal basis. But if we want to go beyond this trivial case and assume that the frame is indeed overcomplete then the correlation $|\langle f_k, f_l\rangle|$ will strongly depend on the redundancy of the frame, which can be thought of as a "measure of overcompleteness". Clearly, the smaller the redundancy the smaller we expect $|\langle f_k, f_l\rangle|$ to be. In $\mathbb{E}^m$ the redundancy ρ of a frame $\{f_k\}_{k=1}^N$ is defined by $\rho = \frac{N}{m}$.

Definition 2.1 *For a given unit norm frame $\{f_k\}_{k=1}^N$ in $\mathbb{E}^m$ we define the* maximal frame correlation $\mathcal{M}(\{f_k\}_{k=1}^N)$ *by*

$$\mathcal{M}(\{f_k\}_{k=1}^N) = \max_{k,l,k\neq l}\{|\langle f_k, f_l\rangle|\}. \tag{10}$$

The restriction to unit norm frames in the definition above is just for convenience, alternatively we could consider general frames and normalize the inner product in (10) by the norm of the frame elements. Hence without loss of generality we can assume throughout this section that all frames are unit norm.

Definition 2.2 *A sequence of vectors $\{u_k\}_{k=1}^N$ in $\mathbb{E}^m$ is called a* Grassmannian frame *if it is the solution to*

$$\min\left\{\mathcal{M}(\{f_k\}_{k=1}^N)\right\}, \tag{11}$$

where the minimum is taken over all unit norm frames $\{f_k\}_{k=1}^N$ in $\mathbb{E}^m$.

In other words a Grassmannian frame minimizes the maximal correlation between frame elements among all unit norm frames which have the same redundancy. Obviously the minimum in (11) depends only on the parameters N and m.

A trivial example for Grassmannian frames in $\mathbb{E}^m$ is to take the n-th roots of unity as frame elements. All the frames generated in this way are unit norm tight, however only for $n = 2$ and $n = 3$ we get equiangular frames. For $n = 2$ we obtain an orthonormal basis and for and for $n = 3$ we obtain the well known tight frame appearing in Chapter 3 of [11].

Two problems arise naturally when studying finite Grassmannian frames:
Problem 1: Can we derive bounds on $\mathcal{M}(\{f_k\}_{k=1}^N)$ for given N and m?
Problem 2: How can we construct Grassmannian frames?

The following theorem provides an exhaustive answer to problem 1. The theorem is new in frame theory but actually it only unifies and summarizes results from various quite different research areas.

Theorem 2.3 *Let* $\{f_k\}_{k=1}^N$ *be a frame for* $\mathbb{E}^m$. *Then*

$$\mathcal{M}(\{f_k\}_{k=1}^N) \geq \sqrt{\frac{N-m}{m(N-1)}}. \tag{12}$$

Equality holds in (12) *if and only if*

$$\{f_k\}_{k=1}^N \text{ is an equiangular tight frame.} \tag{13}$$

Furthermore,

$$\text{if } \mathbb{E}=\mathbb{R} \text{ equality in (12) can only hold if } N \leq \frac{m(m+1)}{2}, \tag{14}$$

$$\text{if } \mathbb{E}=\mathbb{C} \text{ then equality in (12) can only hold if } N \leq m^2. \tag{15}$$

Proof: A proof of the bound (12) can be found in [46, 37]. It also follows from Lemma 6.1 in [44]. One way to derive (12) is to consider the non-zero eigenvalues $\lambda_1, \dots, \lambda_m$ of the Gram matrix $R = \{\langle f_k, f_l\rangle\}_{k,l=1}^N$. These eigenvalues satisfy $\sum_{k=1}^m \lambda_k = N$ and also

$$\sum_{k=1}^m \lambda_k^2 = \sum_{k=1}^N \sum_{l=1}^N |\langle f_k, f_l\rangle|^2 \geq \frac{N^2}{m}, \tag{16}$$

see [37, 44]. The bound follows now by taking the maximum over all $|\langle f_k, f_l\rangle|$ in (16) and observing that there are $N(N-1)/2$ different pairs $\langle f_k, f_l\rangle$ for $k, l = 1, \dots, N$ with $k \neq l$.

Equality in (12) implies $\lambda_1 = \dots = \lambda_m = \frac{N}{m}$, which in turn implies tightness of the frame, and also $|\langle f_k, f_l\rangle|^2 = \frac{N-m}{m(N-1)}$ for all k, l with $k \neq l$ which yields the equiangularity (cf. also [37, 8]). Finally the bounds on N in (14), (15) follow from the bounds in Table II of [12]. ❑

As mentioned in passing in the proof unit norm tight frames meet the bound (16) with equality. We call unit norm frames that meet the bound (12) with equality *optimal Grassmannian frames*. The following corollary will be instrumental in the construction of a variety of optimal Grassmannian frames.

Corollary 2.4 *Let $m, N \in \mathbb{N}$ with $N \geq m$. Assume R is a hermitian $N \times N$ matrix with entries $R_{k,k} = 1$ and*

$$R_{k,l} = \begin{cases} \pm\sqrt{\frac{N-m}{m(N-1)}}, & \text{if } \mathbb{E} = \mathbb{R}, \\ \pm i\sqrt{\frac{N-m}{m(N-1)}}, & \text{if } \mathbb{E} = \mathbb{C}, \end{cases} \tag{17}$$

for $k, l = 1, \dots, N; k \neq l$. If the eigenvalues $\lambda_1, \dots, \lambda_N$ of R are such that $\lambda_1 = \cdots = \lambda_m = \frac{N}{m}$ and $\lambda_{m+1} = \cdots = \lambda_N = 0$, then there exists a frame $\{f_k\}_{k=1}^N$ in $\mathbb{E}^m$ that achieves the bound (12).

Proof: Since R is hermitian it has a spectral factorization of the form $R = W\Lambda W^*$, where the columns of W are the eigenvectors and the diagonal matrix Λ contains the eigenvalues of R. Without loss of generality we can assume that the non-zero eigenvalues of R are contained in the first m diagonal entries of Λ. Set $f_k := \sqrt{\frac{N}{m}}\{W_{k,l}\}_{l=1}^m$ for $k = 1, \dots, N$. By construction we have $\langle f_k, f_l \rangle = R_{l,k}$, hence $\{f_k\}_{k=1}^N$ is equiangular. Obviously $\{f_k\}_{k=1}^N$ is tight, since all non-zero eigenvalues of R are identical. Hence by Theorem 2.3 $\{f_k\}_{k=1}^N$ achieves the bound (12). ❑

On the first glance Corollary 2.4 does not seem to make the problem of constructing optimal Grassmannian frames much easier. However by using a link to graph theory and spherical designs we will be able to derive many explicit constructions of matrices having the properties outlined in Corollary 2.4.

While the concept of Grassmannian frames is new in frame theory there are a number of related concepts in other areas of mathematics. Thus it is time to take a quick journey through these areas which will take us from Grassmannian spaces to spherical designs to coding theory.

Packings in Grassmannian spaces:

The *Grassmannian space* $\mathcal{G}(m, n)$ is the set of all n-dimensional subspaces of the space $\mathbb{R}^m$ (usually the Grassmannian space is defined for $\mathbb{R}$ only, although many problems can be analogously formulated for the complex space). $\mathcal{G}(m, n)$ is a homogeneous space isomorphic to $\mathcal{O}(m)/(\mathcal{O}(n) \times \mathcal{O}(m-n))$, it forms a compact Riemannian manifold of dimension $n(m-n)$.

The *Grassmannian packing problem* is the problem of finding the best packing of N n-dimensional subspaces in $\mathbb{E}^m$, such that the angle between any two of these subspaces becomes as large as possible [8, 6]. In other words, we want to find N points in $\mathcal{G}(m, n)$ so that the minimal distance between any two of them is as large as possible. For our purposes we can concentrate on the case $n = 1$. Thus the subspaces are (real or complex) lines through the origin in $\mathbb{E}^m$ and the goal is to arrange N lines such that the angle between any two of the lines becomes as large as possible. Since maximizing the angle between lines is equivalent to minimizing the modulus of the inner product of the unit vectors generating these lines, it is obvious that finding optimal packings in $\mathcal{G}(m, 1)$ is equivalent to finding finite Grassmannian frames (which also motivated the name for this class of frames).

By embedding the Grassmannian space $\mathcal{G}(m, n)$ into a sphere of radius $\sqrt{n(m-n)/m}$ in $\mathbb{R}^d$ with $d = (m+1)m/2 - 1$, Conway, Hardin, and Sloane are able to apply bounds from spherical codes due to Rankin [35] to derive bounds on the maximal angle between N subspaces in $\mathcal{G}(n, m)$ (see the very inspiring paper [8]). For the case $n = 1$ the bound coincides of course with (12).

Spherical codes:

A *spherical code* $\mathcal{S}(m, N, s)$ is a set of N points (code words) on the m-dimensional unit sphere Ω_m, such that the inner product between any two code words is smaller than s, cf. [9]. By placing the points on the sphere as far as possible from each other one attempts to minimize the risk of decoding errors. *Antipodal spherical codes* are spherical codes which contain with each code word w also the code word $-w$. Clearly, the construction of antipodal spherical codes whose N points are as from each other as possible is closely related to constructing Grassmannian frames.

In coding theory the inequality at the right-hand side of (16) is known as *Welch bound*, cf. [46]. Since unit norm tight frames meet (16) with equality, i.e.,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} |\langle f_k, f_l \rangle|^2 = \frac{N^2}{m},$$

unit norm tight frames are known as *Welch bound equality* (WBE) sequences in coding theory[1]. Inequality (16) and the fact that it is met by unit norm tight frames has recently been rediscovered in [3]. WBE sequences have gained new popularity in connection with the construction of spreading sequences for Code-Division Multiple-Access (CDMA) systems [45, 25, 39]. WBE sequences that meet (12) with equality are called maximum WBE

[1]The authors of [45] incorrectly call WBE sequences tight frames.

(MWBE) sequences [46, 39]. While Welch (among other authors) derived the bound (12) he did not give an explicit construction of MWBE sequences.

Spherical designs:

A *spherical t-design*[2] is a finite subset X of the unit sphere Ω_m in $\mathbb{R}^m$, such that

$$|X|^{-1} \sum_{x \in X} h(x) = \int_{\Omega_m} h(x) dw(x), \tag{18}$$

for all homogeneous polynomials $h \in \mathrm{Hom}_t(\mathbb{R}^m)$ of total degree t in m variables, see e.g. [41]. A spherical design measures certain regularity properties of sets X on the unit sphere Ω_m. Another way to define a spherical t-design is by requiring that, for $k = 0, \dots, t$ the k-th moments of X are constant with respect to orthogonal transformations of $\mathbb{R}^m$. Here are a few characterizations of spherical t-designs that make the connection to the aforementioned areas transparent. For details about the following examples we refer to [13]. Let the cardinality of X be N. X is a spherical 1-design if and only if the Gram matrix $R(X)$ of the vectors of X has vanishing row sums. X is a spherical 2-design if it is a spherical 1-design and the Gram matrix $R(X)$ has only two different eigenvalues, namely N/m with multiplicity m, and 0 with multiplicity $N - m$. An antipodal spherical code on Ω_m is a 3-design if and only if the Gram matrix of the corresponding set of vectors has two eigenvalues.

Equiangular line sets and equilateral point sets:

In [33, 12] Seidel et al. consider sets of lines in $\mathbb{R}^m$ and in $\mathbb{C}^m$ having a prescribed number of angles. They derive upper bounds on the number of lines in the case of one, two, and three prescribed angles (in the latter case, one of the angles is assumed to be zero). Most interesting are those line sets that actually meet the upper bound. In [44] van Lint and Seidel consider a similar problem in elliptic geometry. Since the unit sphere in $\mathbb{R}^m$ serves as model for the $m-1$-dimensional elliptic space $\mathbb{E}^{m-1}$ where any elliptic point is represented by a pair of antipodal points in $\mathbb{R}^m$, the construction of equilateral point sets in elliptic geometry is of course equivalent to the construction of equiangular lines sets in Euclidean geometry. Recall that optimal Grassmannian frames are equiangular, hence the search for equiangular line sets is closely related to the search for optimal Grassmannian frames.

Characterization of strongly regular graphs:

[2] A *spherical* t-design should not be confused with an "ordinary" t-design.

Graphs with a lot of structure and symmetry play a central role in graph theory. Different kinds of matrices are used to represent a graph, such as the Laplace matrix or adjacency matrices [4]. What structural properties can be derived from the eigenvalues depends on the specific matrix that is used. The Seidel adjacency matrix A of a graph Γ is given by

$$A_{xy} = \begin{cases} -1 & \text{if the vertices } x, y \in \Gamma \text{ are adjacent,} \\ 1 & \text{if the vertices } x, y \in \Gamma \text{ are non-adjacent,} \\ 0 & \text{if } x = y. \end{cases} \tag{19}$$

If A has only very few different eigenvalues then the graph is (strongly) regular, cf. [4]. The connection to Grassmannian frames $\{f_k\}_{k=1}^N$ that achieve the bound (12) is as follows. Assume that the associated Gram matrix $R = \{\langle f_k, f_l\rangle\}_{k,l=1}^N$ has entries $\pm\alpha$ and 1 at the diagonal. Then

$$A = \frac{1}{\alpha}(R - I) \tag{20}$$

is the adjacency matrix of a regular two-graph [40]. We will make use of this relation in the next section.

2.1 Construction of Grassmannian frames

In this section we present explicit constructions for optimal finite Grassmannian frames. Note that optimal Grassmannian frames do not exist for all choices of m and N (assuming of course that N does not exceed $(m+1)m/2$ or m^2, respectively). For instance there are no 5 vectors in $\mathbb{R}^3$ with maximal correlation $\frac{1}{\sqrt{6}}$. In fact, although the 5 vectors in $\mathbb{R}^3$ that minimize (11) are equiangular, the maximal inner product is $\frac{1}{\sqrt{5}}$ (but not $\frac{1}{\sqrt{6}}$), see [8]. However there exists an optimal Grassmannian frame consisting of six vectors. The frame elements correspond to the (antipodal) vertices of the icosahedron[3] and the maximal correlation achieves indeed the optimal value $\frac{1}{\sqrt{5}}$. We refer to [8] for details about some of these and other examples. On the other hand the 7 vectors in $\mathbb{R}^3$ that minimize (11) yield a unit norm tight frame, but not an equiangular one (which should not come as a surprise since the choice $N = 7$ exceeds the bound $N \leq m(m+1)/2$). Note that for $\mathbb{C}^3$ we can indeed construct 7 lines that achieve the bound (12), see Subsection 2.1.2.

[3]The Grassmannian frame consisting of five vectors is constructed by removing an arbitrary element of the optimal Grassmannian frame consisting of six vectors.

2.1.1 Grassmannian frames and conference matrices

In this subsection we present explicit constructions of Grassmannian frames with low redundancy. We begin with a simple corollary of Theorem 2.3.

Corollary 2.5 *Let $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$ and $N = m+1$. Then $\{f_k\}_{k=1}^N$ is an optimal Grassmannian frame for $\mathbb{E}^m$ if and only if it is a unit norm tight frame.*

Proof: An optimal Grassmannian frame $\{f_k\}_{k=1}^N$ with $N = m+1$ can be easily constructed by taking the vectors to be the vertices of a regular simplex in $\mathbb{E}^m$, cf [8]. Thus by Theorem 2.3 $\{f_k\}_{k=1}^N$ is a unit norm tight frame. On the other hand it was shown in [21] that all unit norm tight frames with $N = m+1$ are equivalent under multiplication of f_k by $\sigma_k U$, where U is a unitary matrix and $\sigma_k = \pm 1$. Since this equivalence relation preserves inner products it follows that any unit norm tight frame $\{f_k\}_{k=1}^N$ with $N = m+1$ achieves the bound (12). ❑

A unit norm tight frame $\{f_k\}_{k=1}^N$ with $N = m+1$ also provides a spherical 1-design, which can be seen as follows. When $N = m+1$ we can always multiply the elements of $\{f_k\}_{k=1}^N$ by ± 1 such that the Gram matrix R has 1 as its main diagonal entries and $-\frac{1}{m}$ elsewhere. Hence the row sums of R vanish and therefore $\{f_k\}_{k=1}^N$ constitutes a spherical 1-design. It is obvious that the Seidel adjacency matrix A of a graph which is constructed from a regular simplex has $A_{kk} = 0$ and $A_{kl} = -1$ for $k \neq l$, which illustrates nicely the relationship between A and R as stated in (19).

The following construction has been proposed in [32, 8]. An $n \times n$ *conference matrix* C has zeros along its main diagonal and ± 1 as its other entries, and satisfies $CC^T = (n-1)I_n$, see [32]. Conference matrices play an important role in graph theory [40]. If C_{2m} is a symmetric conference matrix, then there exist exist $2m$ vectors in $\mathbb{R}^m$ such that the bound (12) holds with $\rho(2m, m) = 1/\sqrt{2m-1}$. If C_{2m} is a skew-symmetric conference matrix (i.e., $C = -C^T$), then there exist exist $2m$ vectors in $\mathbb{C}^m$ such that the bound (12) holds with $\rho(2m, m) = 1/\sqrt{2m-1}$, see Example 5.8 in [12]. The link between the existence of a (real or complex) optimal Grassmannian frame and the existence of a corresponding conference matrix C_{2m} can be easily as seen as follows. Assume that $\{f_k\}_{k=1}^N$ achieves (12) and denote $\alpha := 1/\sqrt{2m-1}$. We first consider the case $\mathbb{E} = \mathbb{R}$. Clearly the entries of the $2m \times 2m$ Gram matrix $R = \{\langle f_k, f_l\rangle\}_{k,l=1}^N$ are $R_{k,l} = \pm\alpha$ for $k \neq l$ and $R_{k,k} = 1$. Hence

$$C := \frac{1}{\alpha}(R - I) \tag{21}$$

is a symmetric conference matrix. For $\mathbb{E} = \mathbb{C}$ we assume that $R_{k,l} = \pm i\alpha$ for $k \neq l$ and $R_{k,k} = 1$. Then

$$C := \frac{1}{i\alpha}(R - I) \tag{22}$$

is a skew-symmetric conference matrix.

The derivations above lead to the following

Corollary 2.6 *(a) Let $N = 2m$, with $N = p^\alpha + 1$ where p is an odd prime number and $\alpha \in \mathbb{N}$. Then there exists an optimal Grassmannian frame in $\mathbb{R}^m$ which can be constructed explicitly.*
(b) Let $N = 2m$, with $m = 2^\alpha$ with $\alpha \in \mathbb{N}$. Then there exists an optimal Grassmannian frame in $\mathbb{C}^m$ which can be constructed explicitly.

Proof: Paley has shown that if $N = p^\alpha + 1$ with p and α as stated above, then there exists a symmetric $N \times N$ conference matrix. Moreover this matrix can be constructed explicitly, see [34, 18]. For the case $N = 2m = 2^{\alpha+1}$ a skew-symmetric conference matrix can be constructed by the following recursion: Initialize

$$C_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{23}$$

and compute recursively

$$C_{2m} = \begin{bmatrix} C_m & C_m - I_m \\ C_m + I_m & -C_m \end{bmatrix}, \tag{24}$$

then it is easy to see that C_{2m} is a skew-symmetric conference matrix.

An application of Corollary 2.4 to both, the symmetric and the skew-symmetric conference matrix respectively, completes the proof. ❏

Hence for instance there exist 50 equiangular lines in $\mathbb{R}^{25}$ with angle $\text{acos}(1/\sqrt{49})$ and 128 equiangular lines in $\mathbb{C}^{64}$ with angle $\text{acos}(1/\sqrt{127})$. The construction in (23), (24) is reminiscent of the construction of Hadamard matrices. Indeed, $C_m + I_m$ is a skew-symmetric Hadamard matrix.

These constructions yield Grassmannian frames with modest redundancy, in the next two subsections we consider Grassmannian frames with considerably larger redundancy.

2.1.2 Harmonic Grassmannian frames

The elements of a harmonic tight frame $\{f_k\}_{k=1}^N$ for $\mathbb{C}^m$ are given by

$$f_k = \{\omega_l^k\}_{l=1}^m, \tag{25}$$

where the ω_l are distinct N-th roots of unity. Since harmonic tight frames have a number of nice properties [21, 7], it is natural to ask if there exist harmonic Grassmannian frames (beyond the trivial cases $N = m$ and $N = m + 1$). The following example derived by H. König [30] in connection with spherical designs provides an affirmative and constructive answer to this question.

Let p be a prime number and set $m = p^l + 1$ for $l \in \mathbb{N}$ and $N = m^2 - m + 1$. Then there exist integers $0 \le d_1 < \cdots < d_m < N$ such that all numbers $1, \ldots, N-1$ occur as residues mod N of the $n(n-1)$ differences $d_i - d_j, i \ne j$. For $k = 1, \ldots, N$ we define

$$f_k := \Big\{\frac{1}{\sqrt{m}} e^{2\pi i k d_j/N}\Big\}_{j=1}^{m}. \tag{26}$$

It follows immediately from Proposition 4 in [30] that the functions $\{f_k\}_{k=1}^N$ form a harmonic tight Grassmannian frame with $\mathcal{M}(\{f_k\}_{k=1}^N) = \frac{\sqrt{m-1}}{m}$.

2.1.3 Nearly optimal Grassmannian frames

Theorem 2.3 gives an upper bound on the cardinality of optimal Grassmannian frames. If the redundancy of a frame is too large then equality in (12) cannot be achieved. But it is possible to design Grassmannian frames whose cardinality slightly exceeds the bounds in Theorem 2.3, while their maximal correlation is close to the optimal value. For instance for any $m = p^k$ where p is a prime and $k \in \mathbb{N}$ there exist frames $\{f_k\}_{k=1}^N$ in $\mathbb{C}^m$ where $N = m^2 + 1$, with maximal correlation $\mathcal{M} = \frac{1}{\sqrt{m}}$. In fact, these nearly optimal Grassmannian frames are unions of orthonormal bases (and thus form a unit norm tight frame). The modulus of the inner products between frame elements takes on only the values 0 and $\frac{1}{\sqrt{m}}$. We refer to [5, 47] for details about these amazing constructions. They find an important application in quantum physics [47] as well as in the design of spreading sequences for CDMA [25].

Example: Here is an example of a finite Gabor frame that is a nearly optimal Grassmannian frame in $\mathcal{H} = \mathbb{C}^m$ (see [15, 22] for generalities about finite and infinite Gabor frames). Let m be a prime number ≥ 5 and set $g(n) = e^{2\pi i n^3/m}$ for $n = 0, \ldots, m-1$. Then the frame $\{g_{k,l}\}_{k,l=0}^{m-1}$, where

$$g_{k,l}(n) = g(n-k)e^{2\pi i l n/m}, \quad k, l = 0, \ldots, m-1, \tag{27}$$

satisfies $|\langle g_{k,l}, g_{k',l'}\rangle| \in \{0, 1/\sqrt{m}\}$ for all $g_{k,l} \neq g_{k',l'}$, which follows from basic properties of Gaussian sums (cf. [31] and Theorem 2 in [1]). Hence $\mathcal{M}(\{g_{k,l}\}_{k,l=0}^{m-1}) = 1/\sqrt{m}$ while (12) yields $1/\sqrt{m+1}$ as theoretically optimal value. It can be shown that we can add the standard orthonormal basis to the frame $\{g_{k,l}\}_{k,l=0}^{m-1}$ without changing the maximal frame correlation $1/\sqrt{m}$.

3 Infinite-dimensional Grassmannian frames

In this section we extend the concept of Grassmannian frames to frames $\{f_k\}_{k=1}^{\infty}$ in separable infinite-dimensional Hilbert spaces. As already pointed out in Section 2 the maximal correlation $|\langle f_k, f_l\rangle|$ of the frame elements will depend crucially on the redundancy of the frame. While it is clear how to define redundancy for finite frames, it is less obvious for infinite dimensional frames.

The following appealing definition is due to Radu Balan and Zeph Landau [2].

Definition 3.1 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. The redundancy ρ of the frame $\{f_k\}_{k=1}^{\infty}$ is defined as*

$$\rho := \Big(\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K}\langle f_k, S^{-1} f_k\rangle\Big)^{-1}, \tag{28}$$

provided that the limit exists.

Using the concept of ultrafilters Balan and Landau have derived a more general definition of redundancy of frames, which coincides of course with the definition above whenever the limit in (28) exists [2]. In this paper we will restrict ourselves to the definition of redundancy as stated in (28) since it is sufficiently general for our purposes.

Remark: We briefly verify that the definition of frame redundancy by Balan and Landau coincides with our usual understanding of redundancy in some important special cases:
(i) Let $\{f_k\}_{k=1}^{N}$ be a finite frame for an m-dimensional Hilbert space $\mathcal{H}^m$. Let $P : \mathcal{H}^n \to \mathcal{H}^m$ denote the associated projection matrix with entries $P_{k,l} = \langle f_k, S^{-1} f_l\rangle$. We compute

$$\rho = \Big(\frac{1}{N}\sum_{k=1}^{N}\langle f_k, S^{-1} f_k\rangle\Big)^{-1} = \frac{N}{\mathrm{trace}(P)} = \frac{N}{\mathrm{rank}(P)} = \frac{N}{m}, \tag{29}$$

which coincides with the usual definition of redundancy in finite dimensions.

(ii) Let $\{g_{m,n}\}_{m,n\in\mathbb{Z}}$, where $g_{m,n}(x) = g(x-ma)e^{-2\pi inbx}$ be a Gabor frame for $\boldsymbol{L}^2(\mathbb{R})$ with time- and frequency-shift parameters $a, b > 0$. We have from [28] that

$$\langle g_{m,n}, S^{-1}g_{m,n}\rangle = \langle g, S^{-1}g\rangle = ab, \qquad \text{for all } m,n \in \mathbb{Z}, \tag{30}$$

hence $\rho = 1/(ab)$ as expected.

(iii) Assume $\{f_k\}_{k\in\mathcal{I}}$ is a unit norm tight frame. Then $\langle f_k, S^{-1}f_k\rangle = 1/A$ and therefore $\rho = A$, which agrees with the intuitive expectation that for unit norm tight frames the frame bound measures the redundancy of the frame [11].

We need two more definitions before we can introduce the concept of Grassmannian frames in infinite dimensions. In this section $\mathcal{H}$ denotes a separable infinite-dimensional Hilbert space.

Definition 3.2 ([10]) *A* unitary system $\mathcal{U}$ *is a countable set of unitary operators containing the identity operator and acting on* $\mathcal{H}$.

Definition 3.3 *Let $\mathcal{U}$ be a unitary system and Φ be a class of functions for $\mathcal{H}$ with $\|f\|_2 = 1$ for $f \in \Phi$. We denote by $\mathcal{F}(\mathcal{H},\mathcal{U},\Phi)$ the family of frames $\{f_k\}_{k\in\mathcal{I}}$ for $\mathcal{H}$ of fixed redundancy ρ, such that*

$$f_k = U_k f_0, \qquad f_0 \in \Phi, \;\; U_k \in \mathcal{U}, \;\; k \in \mathcal{I}. \tag{31}$$

We say that $\{\varphi_k\}_{k\in\mathcal{I}} \in \mathcal{F}(\mathcal{H},\mathcal{U},\Phi)$ is a Grassmannian frame with respect to $\mathcal{F}(\mathcal{H},\mathcal{U},\Phi)$ if it is the solution of [4]

$$\min_{\{f_k\}_{k\in\mathcal{I}}\in\mathcal{F}(\mathcal{H},\mathcal{U},\Phi)} \Big(\max_{k,l\in\mathcal{I};k\neq l} \big\{ |\langle f_k, f_l\rangle| \big\} \Big) \tag{32}$$

for given ρ.

In the definition above we have deliberately chosen Φ such that it does not necessarily have to coincide with all functions in $\boldsymbol{L}^2(\mathcal{H})$. The reason is that in many applications one is interested in designing frames using only a specific class of functions.

[4]For a frame $\{f_k\}_{k\in\mathcal{I}}$ there always exists $\max\limits_{k\neq l}\{|\langle f_k, f_l\rangle|\}$, otherwise the upper frame bound could not be finite.

In finite dimensions we derived conditions under which Grassmannian frames are also unit norm tight frames. Such a nice and simple relationship does not exist in infinite dimensions. However in many cases it is possible to construct a unit norm tight frame whose maximal frame correlation is close to that of a Grassmannian frame as we will see in the next theorem.

The following definition is due to Gabardo and Han [17].

Definition 3.4 *Let $\mathbb{T}$ denote the circle group. A unitary system $\mathcal{U}$ is called* group-like *if*

$$\text{group}(\mathcal{U}) \subset \mathbb{T}\mathcal{U} := \{tU \;:\; t \in \mathbb{T}, U \in \mathcal{U}\}, \tag{33}$$

and if different $U, V \in \mathcal{U}$ are always linearly independent, where group$(\mathcal{U})$ *denotes the group generated by $\mathcal{U}$.*

Theorem 3.5 *Let $\mathcal{F}(\mathcal{H}, \mathcal{U}, \Phi)$ be given, where $\mathcal{U}$ is a group-like unitary system. For given redundancy ρ assume that $\{\varphi_k\}_{k\in\mathcal{I}}$ is a Grassmannian frame for $\mathcal{F}(\mathcal{H}, \mathcal{U}, \Phi)$ with frame bounds A, B. Then there exists a unit norm tight frame $\{h_k\}_{k\in\mathcal{I}}$ with $h_k = U_k h_0, U_k \in \mathcal{U}$, such that*

$$\max_{k,l\in\mathcal{I};k\neq l} |\langle h_k, h_l\rangle| \le \max_{k,l\in\mathcal{I};k\neq l} |\langle \varphi_k, \varphi_l\rangle| + 2\max\left\{|1-\sqrt{\frac{\rho}{A}}|, |1-\sqrt{\frac{\rho}{B}}|\right\}. \tag{34}$$

Proof: Let S be the frame operator associated with the Grassmannian frame $\{\varphi_k\}_{k\in\mathcal{I}}$. We define the tight frame $\{h_k\}_{k\in\mathcal{I}}$ via $h_k := \sqrt{\rho} S^{-\frac{1}{2}} \varphi_k$. Since $\mathcal{U}$ is a group-like unitary system it follows from (31) above and Theorem 1.2 in [24] that

$$\langle \varphi_k, S^{-1}\varphi_k\rangle = \langle U_k\varphi_0, S^{-1}U_k\varphi_0\rangle = \langle U_k\varphi_0, U_k S^{-1}\varphi_0\rangle = \langle \varphi_0, S^{-1}\varphi_0\rangle, \tag{35}$$

and

$$h_k = \sqrt{\rho} S^{-\frac{1}{2}}\varphi_k = \sqrt{\rho} S^{-\frac{1}{2}} U_k\varphi_0 = \sqrt{\rho} U_k S^{-\frac{1}{2}}\varphi_0. \tag{36}$$

Using Definition 3.1 and (35), we get $\langle \varphi_k, S^{-1}\varphi_k\rangle = \frac{1}{\rho}$ and therefore

$$\|h_k\|^2 = \rho\langle S^{-\frac{1}{2}}\varphi_k, S^{-\frac{1}{2}}\varphi_k\rangle = \rho\langle \varphi_k, S^{-1}\varphi_k\rangle = 1, \qquad \forall k \in \mathcal{I}. \tag{37}$$

Hence $\{h_k\}_{k\in\mathcal{I}}$ is a unit norm tight frame.

We compute

$$\begin{aligned} \Big| |\langle \varphi_k, \varphi_l\rangle| - |\langle h_k, h_l\rangle| \Big| &\le |\langle \varphi_k, \varphi_l\rangle - \langle h_k, h_l\rangle| && (38)\\ &\le |\langle \varphi_k, \varphi_l - h_l\rangle| + |\langle \varphi_k - h_k, h_l\rangle| && (39)\\ &\le \|\varphi_k\|\|\varphi_l - h_l\| + \|h_l\|\|\varphi_k - h_k\|, && (40) \end{aligned}$$

where we have used the triangle inequality and the Cauchy-Schwarz inequality. Note that

$$\|\varphi_l - h_l\| = \|\varphi_l - \sqrt{\rho}S^{-\frac{1}{2}}\varphi_l\| \tag{41}$$
$$\leq \|(I - \sqrt{\rho}S^{-\frac{1}{2}})\|\|\varphi_l\| \tag{42}$$
$$\leq \max\Big\{|1 - \sqrt{\frac{\rho}{A}}|, |1 - \sqrt{\frac{\rho}{B}}|\Big\}. \tag{43}$$

Hence

$$\Big||\langle\varphi_k, \varphi_l\rangle| - |\langle h_k, h_l\rangle|\Big| \leq 2\max\Big\{|1 - \sqrt{\frac{\rho}{A}}|, |1 - \sqrt{\frac{\rho}{B}}|\Big\}, \tag{44}$$

and therefore

$$\max_{k,l\in\mathcal{I};k\neq l}|\langle h_k, h_l\rangle| \leq \max_{k,l\in\mathcal{I};k\neq l}|\langle\varphi_k, \varphi_l\rangle| + 2\max\Big\{|1 - \sqrt{\frac{\rho}{A}}|, |1 - \sqrt{\frac{\rho}{B}}|\Big\}. \tag{45}$$

❏

Remark: (i) Although the canonical tight frame function h_0 does not have to belong to Φ, it is "as close as possible" to the function $\varphi_0 \in \Phi$. Indeed, under the assumptions of Theorem 3.5 the (scaled) canonical tight frame $\{h_k\}_{k\in\mathcal{I}}$ generated by $h_0 = \sqrt{\rho}S^{-\frac{1}{2}}\varphi_0$ minimizes $\|f_0 - \varphi_0\|$ among all tight frames $\{f_k\}_{k\in\mathcal{I}}$ in $\mathcal{F}(\mathcal{H}, \mathcal{U}, L_2(\mathcal{H}))$ (in fact among all possible tight frames), cf. [24] and for the case of Gabor frames [29]. However it is in general not true that $\{h_k\}_{k\in\mathcal{I}}$ also minimizes the maximal frame correlation $\max_{k,l}|\langle f_k, f_l\rangle|$ among all tight frames $\{f_k\}_{k\in\mathcal{I}}$. For instance numerical inspection shows that the tight Gabor frame constructed from the function proposed in [23] (a particular linear combination of Hermite functions) yields a smaller maximal frame correlation than the tight frame canonically associated to the Gaussian.
(ii) If the Grassmannian frame $\{f_k\}_{k\in\mathcal{I}}$ is already tight, then the frame bounds satisfy $A = B = \rho$ and the second term in the right-hand-side of (34) vanishes, as expected.
(iii) Frames that satisfy the assumptions of Theorem 3.5 include shift-invariant frames, Gabor frames, and so-called geometrically uniform frames (see [14] for the latter).

3.1 An example: Grassmannian Gabor frames

In this section we derive Grassmannian frames in $\boldsymbol{L}^2(\mathbb{R})$. We consider Gabor frames in $\boldsymbol{L}^2(\mathbb{R})$ generated by general lattices.

Before we proceed we need some preparation. For $x, y \in \mathbb{R}$ we define the unitary operators of translation and modulation by $T_x f(t) = f(t-x)$, and $M_\omega f(t) = e^{2\pi i\omega t} f(t)$, respectively. Given a function $f \in \boldsymbol{L}^2(\mathbb{R})$ we denote the time-frequency shifted function $f_{x,\omega}$ by

$$f_{x,\omega}(t) = e^{2\pi i\omega t} f(t-x). \tag{46}$$

A lattice Λ of $\mathbb{R}^2$ is a discrete subgroup with compact quotient. Any lattice is determined by its (non-unique) generator matrix $L \in GL(2, \mathbb{R})$ via $\Lambda = L\mathbb{Z}^2$. The volume of the lattice Λ is $\mathrm{vol}(\Lambda) = \det(L)$.

For a function (window) $g \in \boldsymbol{L}^2(\mathbb{R})$ and a lattice Λ in the *time-frequency plane* $\mathbb{R}^2$ we define the corresponding Gabor system $\mathcal{G}(g, \Lambda)$ by

$$\mathcal{G}(g, \Lambda) = \{M_\omega T_x g, \ (x, \omega) \in \Lambda\} \tag{47}$$

Setting $\lambda = (x, \omega)$ we denote $g_\lambda = M_\omega T_x g$. If $\mathcal{G}(g, \Lambda)$ is a frame for $\boldsymbol{L}^2(\mathbb{R})$ we call it a Gabor frame. As in remark (ii) below Definition 3.1 we conclude that the redundancy of $\mathcal{G}(g, \Lambda)$ is $\rho = 1/\mathrm{vol}(\Lambda)$. A necessary but by no means sufficient condition for $\mathcal{G}(g, \Lambda)$ to be a frame is $\mathrm{vol}(\Lambda) \leq 1$, cf. [22]. It is clear that $\max_{\lambda \neq \lambda'} |\langle g_\lambda, g_{\lambda'}\rangle|$ will depend on the volume of the lattice, i.e., on the redundancy of the frame. The smaller the $\mathrm{vol}(\Lambda)$ the larger the $\max_{\lambda \neq \lambda'} |\langle g_\lambda, g_{\lambda'}\rangle|$ is.

One of the main purposes of Gabor frames is to analyze the time-frequency behavior of functions [15]. To that end one employs windows g that are well-localized in time and frequency. The Gaussian $\varphi_\sigma(x) = (2/\sigma)^{\frac{1}{4}} e^{-\pi\sigma x^2}, \sigma > 0$, is optimally localized in the sense that it minimizes the Heisenberg Uncertainty Principle. Therefore Gabor frames using Gaussian windows are of major importance in theory and applications. Our goal is to construct Grassmannian Gabor frames generated by Gaussians. Recall that $\mathcal{G}(\varphi_\sigma, \Lambda)$ is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R})$ whenever $\mathrm{vol}(\Lambda) < 1$, see [22]. Thus in the notation of Definition 3.3 we consider $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R})$, $\mathcal{U} = \{T_x M_y, x, y \in \Lambda$ with $\mathrm{vol}(\Lambda) = \rho\}$, and $\Phi = \{\varphi_\sigma \,|\, \varphi_\sigma(x) = (2\sigma)^{\frac{1}{4}} e^{-\pi x^2/\sigma}, \ \sigma > 0\}$. That means for fixed redundancy ρ we want to find Λ^o among all lattices Λ with $\mathrm{vol}(\Lambda) = \rho$ and φ_σ^o among all Gaussians φ_σ such that

$$\max_{\lambda \neq \lambda'} |\langle (\varphi_\sigma)_\lambda, (\varphi_\sigma)_{\lambda'}\rangle| \tag{48}$$

is minimized.

Since $\hat{\varphi}_\sigma = \varphi_{1/\sigma}$ we can restrict our analysis to Gaussians with $\sigma = 1$, as all other cases can be obtained by a proper dilation of the lattice. To simplify notation we write $\varphi := \varphi_1$.

Since T_x and M_ω are unitary operators there holds $|\langle g_\lambda, g_{\lambda'}\rangle| = |\langle g, g_{\lambda'-\lambda}\rangle|$ for any $g \in \boldsymbol{L}^2(\mathbb{R})$. Furthermore $|\langle \varphi, \varphi_\lambda\rangle|$ is monotonically decreasing with increasing $\|\lambda\|$ (where $\|\lambda\| = \sqrt{|x|^2 + |\omega|^2}$) due to the unimodality, symmetry, and Fourier invariance of φ. These observations imply that our problem reduces to finding the lattice Λ^o of redundancy ρ such that $\max |\langle \varphi, \varphi_\lambda\rangle|$ is minimized where

$$\lambda \in \{Le_1, Le_2, \text{ with } e_1 = [1,0]^T, e_2 = [0,1]^T\}. \tag{49}$$

The ambiguity function of $f \in \boldsymbol{L}^2(\mathbb{R})$ is defined as

$$Af(t,\omega) = \int_{-\infty}^{+\infty} f(x + \frac{t}{2})\overline{f(x - \frac{t}{2})}e^{-2\pi i\omega x}\,dx. \tag{50}$$

There holds $|\langle f, g\rangle|^2 = |\langle Af, Ag\rangle|$ for $f, g \in \boldsymbol{L}^2(\mathbb{R})$. It follows from Proposition 4.76 in [16] that $A\varphi$ is rotation-invariant. Furthermore $A\varphi_\lambda$ is rotation-invariant with respect to its "center" $\lambda = (x, \omega)$ which follows from (4.7) and (4.20) in [22] and the rotation-invariance of $A\varphi$.

Next we need a result from sphere packing theory. Recall that in the classical sphere packing problem in $\mathbb{R}^d$ one tries to find the lattice Λ^o among all lattices Λ in $\mathbb{R}^d$ that solves

$$\max_\Lambda \left\{\frac{\text{Volume of a sphere}}{\text{vol}(\Lambda)}\right\}. \tag{51}$$

For a given lattice Λ the radius r of such a sphere is

$$r = \frac{1}{2}\Big(\min_{\substack{\lambda, \lambda' \in \Lambda \\ \lambda \neq \lambda'}} \{\|\lambda - \lambda'\|\}\Big). \tag{52}$$

Hence solving (51) is equivalent to solving

$$\max_\Lambda\Big\{\min_{\substack{\lambda, \lambda' \in \Lambda \\ \lambda \neq \lambda}} \{\|\lambda - \lambda'\|\}\Big\} \qquad \text{subject to } \text{vol}(\Lambda) = \rho \tag{53}$$

for some arbitrary, but fixed $\rho > 0$. Obviously the minimum has to be taken only over adjacent lattice points.

Due to the rotation-invariance of $A\varphi$ and $A\varphi_\lambda$ and since $A\varphi(x, \omega)$ is monotonically decreasing with increasing (x, ω) we see that the solution of

$$\min_{\Lambda} \max_{\lambda \text{ as in } (49)} \{|\langle \varphi, \varphi_\lambda \rangle|\} \qquad \text{subject to } \mathrm{vol}(\Lambda) = \rho \tag{54}$$

is identical to the solution of (53). It is well-known that the sphere packing problem (53) in $\mathbb{R}^2$ is solved by the hexagonal lattice Λ_{hex}, see [9]. Thus for given redundancy $\rho > 1$ the Gabor frame $\mathcal{G}(\varphi, \Lambda_{hex})$ is a Grassmannian frame, where the generator matrix of Λ_{hex} is given by

$$L_{\mathrm{h}ex} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{\rho}} & \frac{1}{\sqrt[4]{3}\sqrt{2\rho}} \\ 0 & \frac{\sqrt[4]{3}}{\sqrt{2\rho}} \end{bmatrix}. \tag{55}$$

Remark: (i) The above result can be generalized to higher dimensions, since the ambiguity function of a d-dimensional Gaussian is also rotation-invariant. Hence a Grassmannian Gabor frame with Gaussian window is always associated with the optimal lattice sphere packing in $\mathbb{R}^{2d}$. However in higher dimensions explicit solutions to the sphere packing problem are in general not known. [9].
(ii) If we define the Gaussian with complex exponent $\sigma = u + iv$ with $u > 0$ (i.e., chirped Gaussians in engineering terminology) then it is not hard to show that a properly chirped Gaussian associated with a rectangular lattice also yields a Grassmannian Gabor frame.

The Grassmannian Gabor frame constructed above has found application in wireless communications in connection with so-called lattice orthogonal frequency division multiplex (OFDM) systems, see [43]. It has been shown in [43] that Grassmannian Gabor frames can reduce the effect of interchannel interference and intersymbol interference for time-frequency dispersive wireless channels.

4 Erasures, coding, and Grassmannian frames

Recently finite frames have been proposed for multiple description coding for erasure channels, see [21, 20, 7]. We consider the following setup. Let $\{f_k\}_{k=1}^N$

be a frame in $\mathbb{E}^m$. As in (6) and (7) we denote the associated analysis and synthesis operator by T and T^* respectively. Let $f \in \mathbb{E}^m$ represent the data to be transmitted. We compute $y = Tf \in \mathbb{E}^N$ and send y over the erasure channel. We denote the index set that corresponds to the erased coefficients by $\mathcal{E}$ and the surviving coefficients are indexed by the set $\mathcal{R}$. Furthermore we define the $N \times N$ *erasure matrix* Q via

$$Q_{kl} = \begin{cases} 0 & \text{if } k \neq l, \\ 0 & \text{if } k = l \text{ and } k \in \mathcal{E}, \\ 1 & \text{if } k = l \text{ and } k \in \mathcal{R}. \end{cases} \tag{56}$$

Let ε represent additive white Gaussian noise (AWGN) with zero mean and power spectral density σ^2. The data vector arriving at the receiver can be written as $\tilde{y} := Qy + \varepsilon$.

The frame $\{f_k\}_{k=1}^N$ is robust against e erasures, if $\{f_k\}_{k\in\mathcal{R}}$ is still a frame for $\mathbb{E}^m$ for any index set $\mathcal{R} \subset \{0, \ldots, N-1\}$ with $|\mathcal{R}| \geq N - e$. In this case standard linear algebra implies that f can be exactly reconstructed from $\tilde{y}$ in the absence of noise[5].

In general, when we employ a minimum mean squared (MMSE) receiver we compute the (soft) estimate

$$\tilde{f} = (T_{\mathcal{R}}^* T_{\mathcal{R}} + \sigma^2 I_m)^{-1} T_{\mathcal{R}}^* \tilde{y}, \tag{57}$$

where $T_{\mathcal{R}}$ is the analysis operator of the frame $\{f_k\}_{k\in\mathcal{R}}$. This involves the inversion of a possibly large matrix (no matter if noise is present or not) that can differ from one transmission to the next one. The costs of an MMSE receiver may be prohibitive in time-critical applications. Therefore one often resorts to a matched filter receiver which computes the estimate

$$\tilde{f} = cT^* \tilde{y}, \tag{58}$$

where T is the analysis operator of the original frame $\{f_k\}_{k=1}^N$ and $c > 0$ is a scaling constant. The advantage of an MMSE receiver is the better error performance while a matched filter receiver can be implemented at lower computational cost.

[5]For instance the so-called harmonic frames are robust against up to $N-m$ erasures [21], which does not come as a surprise to those researchers who are familiar with Reed-Solomon codes or with the fundamental theorem of algebra.

Robustness against the maximal number of erasures is not the only performance criterion when designing frames for coding. Since any transmission channel is subject to AWGN, it is important that the noise does not get amplified during the transmission process. Yet another criterion is ease of implementation of the receiver. It is therefore natural to assume $\{f_k\}_{k=1}^N$ to be a unit norm tight frame, since in case of no erasures (i) the MMSE receiver coincides with the matched filter receiver and (ii) AWGN does not get amplified during transmission. See [38] for an analysis of MMSE receivers and unit norm tight frames (i.e., WBE sequences).

Our goal in this section is to design a unit norm tight frame such that the performance of a matched filter receiver is maximized in presence of an erasure channel. In other words the approximation error $\|f - \tilde{f}\|$ is minimized, where $\tilde{f}$ is computed via a matched filter receiver, i.e., $\tilde{f} = \frac{m}{N} T^* \tilde{y}$ with $\tilde{y} = Qy + \varepsilon$.

We estimate the reconstruction error via

$$\begin{aligned} \|f - \tilde{f}\| =& \|f - \frac{m}{N} T^*(QTf + \varepsilon)\| && (59) \\ \leq & \frac{m}{N}\|T^*Tf - T^*QTf\| + \frac{m}{N}\|T^*\varepsilon\| && (60) \\ \leq & \frac{m}{N}\|T^*PT\|_2 \|f\| + \sigma, && (61) \end{aligned}$$

where we used the notation $P = I - Q$. Since P is an orthogonal projection and therefore satisfies $PP^* = P$ there holds

$$\|T^*PT\|_2 = \|(PT)^*PT\|_2 = \|PT(PT)^*\|_2 = \|PTT^*P\|_2. \tag{62}$$

Hence we should design our tight unit norm frame $\{f_k\}_{k\in\mathcal{I}}$ such that $\|PTT^*P\|_2$ is minimized, where the minimum is taken over all matrices $P = I - Q$, with Q as defined in (56).

Recall that $TT^* = \{\langle f_l, f_k\rangle\}_{k,l\in\mathcal{I}}$, hence $PTT^*P = \{\langle f_l, f_k\rangle\}_{k,l\in\mathcal{I}'}$. Furthermore

$$\|PTT^*P\|_2 \leq \sqrt{\|PTT^*P\|_\infty \|PTT^*P\|_1} = \max_{k\in\mathcal{I}'} \sum_{l\in\mathcal{I}'} |\langle f_k, f_l\rangle|. \tag{63}$$

This suggests to look for frames for which $\max\limits_{k,l,k\neq l} |\langle f_k, f_l\rangle|$ is minimized. In other words we should look for Grassmannian frames.

Remark: (i) In case of one erasure it has been shown in [21] (in case of unknown σ) that unit norm tight frames are optimal with respect to minimizing

the influence of AWGN when using the MMSE receiver, cf. also [14]. In case of one erasure unit norm tight frames also minimize the reconstruction error when using a matched filter receiver.
(ii) Holmes and Paulsen have shown that Grassmannian frames are optimal with respect to up to two erasures [27]. This can be easily seen by minimizing the operator norm of the matrix PTT^*P, which in this case reduces *exactly* to the problem of minimizing $\max|\langle f_k, f_l\rangle|$ for all k, l with $k \neq l$.
(iii) There is strong numerical evidence that the optimal Grassmannian frames of part (b) in Corollary 2.6 are even optimal for three erasures (however this is not the case for the frames constructed in part (a)).
(iv) Grassmannian frames are in general not robust against $N - m$ erasures if $N > m + 2$.

Example: We elaborate further an example given in [21], where the authors consider the design of multiple description coding frames $\{f_k\}_{k=1}^N$ in $\mathbb{E}^m$ with $m = 3$ and $N = 7$. As in Examples 4.2 and 4.3 in [21] we consider an erasure channel with AWGN, but unknown noise level. Without knowledge of σ the reconstruction formula of the MMSE receiver simplifies to $\tilde{f} = (T_{\mathcal{R}}^* T_{\mathcal{R}})^{-1} T_{\mathcal{R}}^* \tilde{y}$. Standard numerical analysis tells us that the smaller the condition number of $T_{\mathcal{R}}^* T_{\mathcal{R}}$ the smaller the amplification of the noise in the reconstruction [19]. We therefore compare the condition number of different unit norm tight frames for $m = 3, N = 7$ after up to four frame elements have been randomly removed.

We consider three types of unit norm tight frames. The first frame is an optimal complex-valued Grassmannian frame. Its vectors $f_1, \ldots, f_7$ are given by

$$f_k := \frac{1}{\sqrt{3}} \left\{ e^{2\pi i k d_j / 7} \right\}_{j=1}^3, \qquad k = 1, \ldots, 7, \tag{64}$$

where $d_j \in \{0, 1, 5\}$, see also Subsection 2.1.2. The second frame is constructed by taking the first three rows of a 7×7 DFT matrix and using the columns of the resulting (normalized) 3×7 matrix as frame elements (this is also called a harmonic frame in [21]). The last frame is a randomly generated unit norm tight frame. Since all three frames are unit norm tight, they show identical performance for one random erasure, the condition number of the frame operator in this case is constant 1.322. Since the frames are of small size, we can easily compute the condition number for all possible combinations of two, three, and four erasures. We then calculate the maximal and mean average condition number for each frame. As can be seen from the results in Table 4 the optimal Grassmannian frame outperforms the other two frames in all cases, except for the average condition number for two erasures, where its condition number is slightly larger. This example demonstrates the potential of Grassmannian frames for multiple description coding.

	2 erasures		3 erasures		4 erasures	
Cond.number	mean	max	mean	max	mean	max
Grassmannian frame	1.645	1.645	2.045	2.189	3.056	3.635
DFT-submatrix frame	1.634	1.998	2.199	3.602	4.020	8.589
Random unit norm tight	1.638	1.861	2.095	3.792	3.570	12.710

Table 1: Comparison of mean average and maximal condition number of frame operator in case of two, three, and four random erasures. We compare an optimal Grassmannian frame, a DFT-based unit norm tight frame, and a random unit norm tight frame. The Grassmannian frame shows the best overall performance.

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